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A NOTE ON CRITICAL REFLECTIONS OF WAVES IN AN ISOTROPIC ELASTIC PLATE†

R. K. KAUL‡ and G. HERRMANN
Stanford University, Stanford, CA 94305, U.S.A.

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Abstract—At grazing incidence of a flexural P-wave, or at grazing incidence of an extensional SV-wave, the wave motion vanishes in the case of an isotropic, elastic plate with traction-free surfaces. However, it is shown that the wave motion in these critical cases can be obtained from the evanescent Rayleigh–Lamb solution, by using d'Alembert's limiting procedure.

In a Cartesian co-ordinate system, consider an isotropic, elastic plate of thickness $2b$, with major faces $x_2 = \pm b$ parallel to the (x_1, x_3)-middle plane and of infinite extent otherwise. Let λ and μ be the Lamé's constants and ρ the mass density of the homogeneous, isotropic, elastic material. In a state of plane strain, the two-dimensional motion of the plate is described by the two wave equations

$$\begin{aligned} \nu_p{}^2\nabla^2\varphi &= \ddot{\varphi}, \\ \nu_s{}^2\nabla^2\psi &= \ddot{\psi}, \end{aligned} \tag{1}$$

where $\nu_p{}^2 = (\lambda + 2\mu)/\rho$, $\nu_s{}^2 = \mu/\rho$, ∇^2 is the two-dimensional Laplacian operator and differentiation with respect to time is indicated by superposed dot. The two Cartesian components of the displacement u_1 and u_2 in terms of the displacement potentials φ and ψ are given by

$$\begin{aligned} u_1 &= \partial\varphi/\partial x_1 + \partial\psi/\partial x_2, \\ u_2 &= \partial\varphi/\partial x_2 - \partial\psi/\partial x_1, \\ u_3 &= \text{const.} \end{aligned} \tag{2}$$

For wave-motion which is harmonic in the x_1-direction and anti-symmetric with respect to the middle plane of the plate, the appropriate solution of eqn (1) is

$$\begin{aligned} \varphi &= A \sin \alpha x_2 \operatorname{ixp}(\xi x_1 - \omega t), \\ \psi &= iD \cos \beta x_2 \operatorname{ixp}(\xi x_1 - \omega t), \end{aligned} \tag{3}$$

where i is the fourth root of unity in the Argand plane, and for convenience $\operatorname{ixp}(\ldots.) \equiv \exp i(\ldots.)$. In this case the wave normals of the P-wave and the SV-wave make with the positive x_2-axis, the angles of $\theta_1 = \pm \arctan \xi/\alpha$ and $\theta_2 = \pm \arctan \xi/\beta$, respectively. In an unbounded medium, the circular frequency ω in rad sec, the two wave numbers α and β in the thickness direction, and the wave number ξ in the x_1-direction, are related to each other by the equations

$$\begin{aligned} \kappa\alpha &= (\omega^2/\nu_s{}^2 - (\kappa\xi)^2)^{1/2}, \\ \beta &= (\omega^2/\nu_s{}^2 - \xi^2)^{1/2}, \end{aligned} \tag{4}$$

where $\kappa^2 = (\nu_p/\nu_s)^2 = 2(1-\nu)/(1-2\nu)$.

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‡On leave of absence from State University of New York, Buffalo, NY 14214, U.S.A.

From eqns (2) and (3) the displacements antisymmetric with respect to the middle plane are

$$u_1 = i(A\xi \sin \alpha x_2 - D\beta \sin \beta x_2) \operatorname{ixp} \xi x_1,$$
$$u_2 = (A\alpha \cos \alpha x_2 + D\xi \cos \beta x_2) \operatorname{ixp} \xi x_1, \tag{5}$$

where time dependence has been suppressed for convenience. From the displacement field (5), the stress field can easily be computed and in plane strain it is given by[1, 2]

$$\tau_{22} = \mu[A(\xi^2 - \beta^2) \sin \alpha x_2 - 2D\xi\beta \sin \beta x_2] \operatorname{ixp} \xi x_1,$$
$$\tau_{21} = i\mu[2A\alpha\xi \cos \alpha x_2 + D(\xi^2 - \beta^2) \cos \beta x_3] \operatorname{ixp} \xi x_1, \tag{6}$$
$$\tau_{11} = \mu[A(2\alpha^2 - \beta^2 - \xi^2) \sin \alpha x_2 + 2D\xi\beta \sin \beta x_2] \operatorname{ixp} \xi x_1.$$

For the two major faces of the plate to be free of traction, $\tau_{21} = \tau_{22} = 0$ on $x_2 = \pm b$. From eqn $(6)_{1,2}$ we therefore get for flexural motion

$$\frac{D}{A} = \frac{(\xi^2 - \beta^2) \sin \alpha b}{2\xi b \sin \beta b} = -\frac{2\xi\alpha \cos \alpha b}{(\xi^2 - \beta^2) \cos \beta b}, \tag{7}$$

and the frequency equation in this case is given by

$$(\xi^2 - \beta^2)^2 \sin \alpha b \cos \beta b + 4\xi^2 \alpha\beta \cos \alpha b \sin \beta b = 0. \tag{8}$$

The results obtained so far are well known and the details of the frequency spectrum are well discussed by Mindlin[2]. However, in this note we are concerned with certain critical situations in which the frequency eqn (8) vanishes identically. Thus consider the case of grazing incidence of P-wave when $\theta_1 = \pm \pi/2$. For non-zero values of propagation constant ξ, this corresponds to the case

$$\alpha = 0,$$
$$\beta = (\kappa^2 - 1)^{1/2} \xi, \tag{9}$$
$$\omega/\xi = v_p.$$

In this case frequency eqn (8) vanishes identically, the amplitude coefficient $D/A = 0$, and from eqn (5) we see that the flexural motion vanishes altogether. It can similarly be shown that the extensional motion ceases altogether in the case of grazing incidence of SV-wave, corresponding to $\theta_2 = \pm \pi/2$, $\beta = 0$.

These two cases are not isolated examples of evanescent wave motion at grazing incidence of P-wave and SV-wave in a plate, but is a general phenomenon in a large class of problems involving wave motion when the wave number changes from real to imaginary values. Such critical reflections of elastic waves at free surface of a semi-infinite medium were first-examined by Goodier and Bishop[3], but a similar phenomenon in the case of plates, cylinders, shells, etc., seems to have been overlooked in the literature. Our aim in this brief note is to show that in such critical cases, a non-trivial solution can easily be obtained from the evanescent solution by using a simple procedure due to d'Alembert.

In the present problem the wave numbers α and β are both real for $\omega > v_p\xi$; α is imaginary and β is real for $v_s\xi < \omega < v_p\xi$; and α and β are both imaginary for $\omega < v_s\xi$. Thus the change from real to imaginary values of α takes place at the phase velocity v_p and a similar transition in β occurs at the phase velocity v_s. Thus in eqn (3), when α changes to $i\alpha$, the trigonometric function $\sin \alpha x_2$ changes to $\sinh \alpha x_2$ and when β changes to $i\beta$, the two functions $\sin \alpha x_2$, $\cos \beta x_2$ change to $\sinh \alpha x_2$, $\cosh \beta x_2$, respectively. However, at the critical value of $\alpha = 0$, the amplitude ratio D/A is still zero. This clearly suggests that the form of the solution (3), does not allow a uniform change from real to imaginary wave numbers and needs modification. Indeed, it is now obvious that when $\alpha = 0$, $(3)_1$ is not an appropriate solution of the differential equation $(1)_1$, because in this case the two linearly independent solutions merge into a common solution, and for a second order differential equation we need one more solution which is linearly independent of the confluent solution. In this connection, it has been pointed out by Jardetzky [4], that the two linearly independent solutions have to be taken in different forms for the grazing and non-grazing incidence. Using Jardetzky's idea, it is thus possible to find the two linearly independent solutions in our limiting case of grazing incidence. However, our aim in this note is to show that in the critical case of grazing incidence, when the two solutions merge into a common solution, the two linearly independent solutions can be directly retrieved from the general solutions (3)–(8), by using an elegant and powerful method developed by d'Alembert [5]. Thus following d'Alembert, the solution in the critical case when $\alpha = 0$, can be obtained from eqn $(3)_1$, if we first differentiate $\sin \alpha x_2$ with respect to α and then take the limit as $\alpha \to 0$. We therefore replace $\sin \alpha x_2$ by the limiting value $(\partial \sin \alpha x_2/\partial \alpha)_{\alpha\to 0} = x_2$. From eqn $(9)_2$ we see that when $\alpha = 0$, $\beta = (\kappa^2-1)^{1/2}\xi$ and following d'Alembert's procedure we get from eqn (3) the non-trivial solution

$$\begin{aligned} \varphi &= A x_2 \operatorname{ixp} \xi x_1, \\ \psi &= iD \cos(\kappa^2-1)^{1/2}\xi \operatorname{ixp} \xi x_1. \end{aligned} \tag{10}$$

Again, $(\partial\alpha \cos \alpha x_2/\partial\alpha)_{\alpha\to 0} = 1$ and therefore from eqn (5) we obtain the non-trivial displacement field

$$\begin{aligned} u_1 &= i(A\xi x_2 - D(\kappa^2-1)^{1/2}\xi \sin(\kappa^2-1)^{1/2}\xi x_2) \operatorname{ixp} \xi x_1, \\ u_2 &= (A + D\xi \cos(\kappa^2-1)^{1/2}\xi x_2) \operatorname{ixp} \xi x_1, \\ \partial u_1/\partial x_1 + \partial u_2/\partial x_2 &= -A\xi^2 x_2 \operatorname{ixp} \xi x_1. \end{aligned} \tag{11}$$

From eqn (6), the non-trivial stresses in the critical case are

$$\begin{aligned} \tau_{22} &= \mu\xi^2[A(2-\kappa^2)x_2 - 2D(\kappa^2-1)^{1/2} \sin(\kappa^2-1)^{1/2}\xi x_2] \operatorname{ixp} \xi x_1, \\ \tau_{21} &= i\mu\xi[2A + D\xi(2-\kappa^2)\cos(\kappa^2-1)^{1/2}\xi x_2] \operatorname{ixp} \xi x_1, \\ \tau_{11} &= \mu\xi^2[-A\kappa^2 x_2 + 2D(\kappa^2-1)^{1/2} \sin(\kappa^2-1)^{1/2}\xi x_2] \operatorname{ixp} \xi x_1, \\ \tau_{11} + \tau_{22} &= -2(\lambda+\mu)A\xi^2 x_2 \operatorname{ixp} \xi x_1. \end{aligned} \tag{12}$$

The frequency equation at the turning point can also be directly obtained from eqn (8), by differentiating it first with respect to α and then taking the limit as $\alpha \to 0$.† For antisymmetric modes, we thus obtain from eqn (8) the critical form

$$\tan(\kappa^2-1)^{1/2}\xi b + \frac{(\kappa^2-2)^2}{4(\kappa^2-1)^{1/2}}\xi b = 0, \quad \xi b \geq 0. \tag{13}$$

We now define a non-dimensional frequency $\Omega = \omega/\omega_s$, where $\omega_s = \pi v_s/2b$ is the lowest antisymmetric thickness-shear frequency. But from $(9)_3$, $\omega/v_s = \kappa\xi$ and therefore $\xi b = (\pi/2)\sigma\Omega$ where $\sigma \equiv 1/\kappa$. Thus in terms of the non-dimensional frequency Ω, the limiting form of the frequency equation (13) for the critical flexural motion can be written in the form

$$\tan\frac{\pi}{2}(1-\sigma^2)^{1/2}\Omega + \frac{\pi(1-2\sigma^2)^2}{8\sigma^2(1-\sigma^2)^{1/2}}\Omega = 0, \tag{14}$$

where $\sigma \equiv 1/\kappa < 1$ and $\Omega \geq 0$.

The zeros of this transcendental equation can easily be computed and for $\nu = 0{\cdot}31$, the first

†For a turning point theorem see Appendix.

few zeros are $\Omega_0 = 0$ and

$$
\begin{aligned}
&\Omega_1 = 1{\cdot}918987, \quad \Omega_2 = 4{\cdot}000277, \quad \Omega_3 = 6{\cdot}206215,\\
&\Omega_4 = 8{\cdot}474232, \quad \Omega_5 = 10{\cdot}772959, \quad \Omega_6 = 13{\cdot}088257, \qquad (15)\\
&(1-\sigma^2)^{1/2}\Omega_n \approx \{(2n-1) + 16\sigma^3(1-\sigma^3)/[(2n-1)\pi^3(1-2\sigma^3)^3]\}.\\
&n = 7, 8, 9, \ldots
\end{aligned}
$$

These zeros are additional points on the Rayleigh–Lamb frequency spectrum; they are the points of intersection of the flexural branches of the spectrum with the line representing the dilatational phase velocity in an infinite medium. This corresponds to the intersections of the dotted curves with the line *OD* in Fig. 19 of Ref. [2] or Fig. (3) of Ref. [1]. Exact values of these additional zeros, and their locations with respect to the grid of bounds, are quite useful in sketching the intricate spectrum.

A similar analysis can be carried out for extensional waves when $\beta = 0$, that is, for grazing incidence of SV-wave. The limiting form of the frequency equation in this case is given in Ref. [1] (eqn (2.34)) and in the present notation it is equivalent to

$$\tanh\frac{\pi}{2}(1-\sigma^2)^{1/2}\Omega = \frac{\pi\Omega}{8(1-\sigma^2)^{1/2}}, \quad \Omega \geq 0. \qquad (16)$$

For $\sigma < 1$, corresponding to $\nu = 0{\cdot}31$, the equation has only one real zero located on the line *OE* in Fig. 19 of Ref. [2]. In this case the asymptotic value of the root is given by

$$\Omega_0 \approx \frac{4}{\pi\gamma^{1/2}}\left[1 - \frac{\gamma}{-1 + \dfrac{\gamma}{+1 - \tanh(1/\gamma)}}\right], \qquad (17)$$

where $\gamma^{-1} \equiv 4(1-\sigma^2)$. For $\nu = 0{\cdot}31$, the asymptotic value is given by $\Omega_0 = 2{\cdot}154348$ which compares favourably with the exact value $2{\cdot}154098$.

We may also note in passing that such a critical situation does not arise for grazing incidence of flexural SV-wave, or for grazing incidence of extensional P-wave. It can easily be verified that in each of these two cases the motion is not ethereal and therefore the desired solution in each case can easily be obtained by the usual limiting procedure. This standard limiting procedure will not yield motion of the type (10) and (11) exhibiting linear variation in the thickness direction.†

It is interesting to note that in the (Ω, ξ)-plane, the spectral lines exhibit an inflection at these turning points, suggesting that the group velocity has a stationary value. This may provide useful information in ultrasonic delay line devices. In conclusion we may remark that the ideas developed in this note, of retrieving the solution at grazing incidence, from the general solution, have many applications, some of which will be the subject of forthcoming papers on composites with periodic structure.

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APPENDIX

The derivation of frequency eqn (13) from the general frequency eqn (8), when $\alpha \to 0$, is based on the following simple *turning point* theorem.

†See Ref. [2], pp. 217–218.

Theorem. *Let the determinantal frequency equation* $\Delta(\Omega; \alpha_1, \alpha_2, \ldots, \alpha_m, \ldots, \alpha_n) = 0$ *be a* holomorphic *function of one real and n complex variables* $\{\Omega; \alpha_1, \alpha_2 \ldots, \alpha_m, \ldots, \alpha_n\}$. *If* $\Delta(\Omega; \alpha_1, \alpha_2, \ldots, \alpha_m, \ldots, \alpha_n)$ *vanishes* identically *when* m *complex variables* $\alpha_1 = \alpha_2 = \ldots = \alpha_m \to 0$, $(m \leq n)$, *then the frequency equation at the* turning point *is given by*

$$\partial_{\alpha_1}\partial_{\alpha_2}\ldots\partial_{\alpha_m}\Delta(\Omega; \alpha_1, \alpha_2, \ldots, \alpha_m, \ldots, \alpha_n) = 0,$$

$$\operatorname{Lim} \alpha_1 = \alpha_2 = \ldots = \alpha_m \to 0; \; m \leq n. \tag{A1}$$

where $\partial_{\alpha_i} \equiv \partial/\partial\alpha_i$.

The proof of this theorem follows from the properties of holomorphic functions and use of de l'Hôpital's rule.